MACROECONOMIC GAMES ON TIME SCALES

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ABSTRACT. We use simple concepts and results of time scales calculus to allow for more dynamics in games - focusing on those with inefficient equilibria. We first allow players to move on discrete homogeneous time scales to capture the fact that economic agents' decisions may be infrequent. We then examine arbitrary heterogeneous time scales to allow for time-varying frequency as well as probabilistic actions. In both cases we derive conditions under which the inefficient outcome is eliminated from the set of possible equilibria, i.e. under which efficiency is guaranteed.

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1. INTRODUCTION

The second half of the 20th century witnessed a surge in the interest in difference equations and applications. This unprecedented wave has been mainly driven by the need for a comprehensive discrete-time calculus coming from two rapidly developing areas of science. First, numerical analysis arose to connect the continuous world of mathematical analysis and the discrete world of computer science. Second, social sciences required simple tools to examine various real-world phenomena, which are discrete in nature.

Over time however, it has become evident that discrete calculus often leads to dramatically different results from its continuous counterpart. This has given rise to research that tried to build the missing bridges and understand the differences. Among this trend, time scales calculus proposed by Hilger [4] emerged as a simple and remarkably general unifying tool (for a detailed treatment see Bohner, Peterson [2] and [3]).

In this paper, we show how time scale calculus can be used to generate novel insights in economics by generalizing the timing structure of games. Specifically, we allow for richer dynamics than the standard repeated games or alternating move games (see Maskin, Tirole [8] and Figure 1). As an example, we use the influential 'time inconsistency' game first described by Kydland, Prescott [5] that identified a serious problem of discretionary policymaking (for a different macroeconomic application, see Libich, Stehlik [6]).

We first summarize the properties and consequences of the discrete time setting of a standard repeated game and then generalize these by including more discrete dynamics using homogeneous time scales. In the last section, we also consider cases with continuous time, heterogenous players, time-varying frequencies and probability models making use of heterogeneous time scales.

This paper is intended for a mathematical audience; for more details on the game theoretic and macroeconomic aspects see Libich, Stehlik [7].

While some of the reported results can be proven without using time scales, our analysis demonstrates the basic advantages of this tool - simplicity and generality.

2. TIME SCALES - BASIC NOTATIONS

A time scale is defined as an arbitrary nonempty closed subset of $\mathbb{R}$. In order to classify different classes of points on time scales, we introduce the forward (backward) jump operator $\sigma(t)$ at t for $t < \sup \mathbb{T}$ ($\rho(t)$ at $t \in \mathbb{T}$ for $t > \inf \mathbb{T}$), defined by

$$\sigma(t) = \inf\{\tau > t : \tau \in \mathbb{T}\}, \quad (\rho(t) = \sup\{\tau < t : \tau \in \mathbb{T}\},)\ .$$

If $\mathbb{T} = \mathbb{R}$ then $\sigma(t) = t = \rho(t)$. If $\mathbb{T} = \mathbb{Z}$ then $\sigma(t) = t + 1$ and $\rho(t) = t - 1$.

One can define the so-called delta derivative y^Δ so that (see [2] for more details).

$$y^\Delta = \begin{cases} y' & \text{if } \mathbb{T} = \mathbb{R} \\ \Delta y & \text{if } \mathbb{T} = \mathbb{Z}, \end{cases}$$

and its inverse, delta (Cauchy, Newton, Lebesgue) integral so that

$$\int\limits_a^b f(t)\Delta t = \begin{cases} \int\limits_a^b f(t)\mathrm{d}t & \text{if} \quad \mathbb{T} = \mathbb{R}, \\ \sum\limits_{t=a}^{b-1} f(t) & \text{if} \quad \mathbb{T} = \mathbb{Z}. \end{cases}$$

3. THE TIME INCONSISTENCY GAME

We follow the standard version of the time inconsistency game elaborated by Barro, Gordon [1]. There are two players, the policymaker g and the public p, whose instruments are inflation, π, and wage inflation, w, respectively. For simplicity, both are patient, i.e. they do not discount the future. Each player can select from two options: a low (L) and a high (H) level. In its general form, the game can be summarized in the payoff matrix

		Public	
		L	H
Government	L	a,v	b,x
	H	c,y	d,z

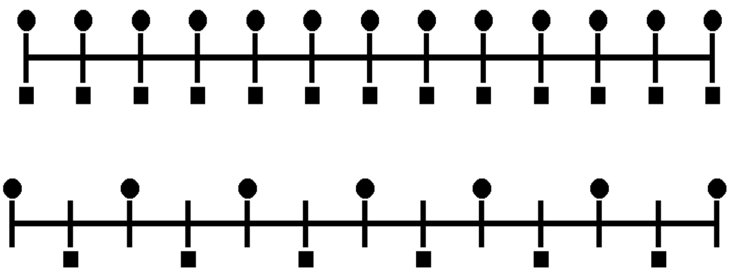

FIGURE 1. Standard repeated and alternating move game settings - timing of moves with two players.

where the payoffs $a, \ldots, z$ satisfy the following constraints (for how these are derived from the standard Barro and Gordon model see Libich, Stehlik [7]).

$$c > a = 0 > d > b, c = -d = -\frac{b}{2} \text{ and } q > v, q \geq z > x. \tag{1}$$

For illustration we will also use the following specific payoffs

$$c = 1 > a = 0 > d = -1 > b = -2 \text{ and } q = z = 0 > v = x = -1, \tag{2}$$

which transforms the above general game into:

		Public	
		L	H
Government	L	0,0	-1,-1
	H	$\frac{1}{2}, -1$	$-\frac{1}{2}, 0$

The standard one shot game has a unique Nash equilibrium, (H, H), which is however inefficient - it is Pareto dominated by the non-Nash (L, L) outcome. Intuitively, the policymaker is tempted to create surprise inflation in order to boost output and reduce unemployment. Since the public is rational, however, it will expect higher inflation and both players will be worse off.

4. DISCRETE HOMOGENEOUS TIME SCALES

4.1. **Assumptions.** In extending the timing structure of the game we adopt all the assumptions of a standard (finitely) repeated game.

First, the game starts with a simultaneous move. Second, the time intervals of actions, $r^g \in \mathbb{N}$ and $r^p \in \mathbb{N}$, are constant and common knowledge. Third, the game finishes after T periods, where $T \in \mathbb{N}$ denotes the least common multiple of r^g and r^p (under the conditions we are interested in, further repetition will not alter the outcome). Fourth, players are rational, have common knowledge of rationality and complete information about the structure of the game, opponents' payoffs, and can observe all past periods' moves (games of perfect monitoring).

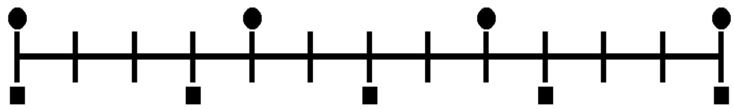

FIGURE 2. An example of a setting with homogeneous time scales, $r^g = 4$ and $r^p = 3$.

In other words, the assumptions define three time scales, one for the policymaker, one for the public and one for the game (see Figure 2)

$$\mathbb{T}_g = \{0, r^g, 2r^g, \dots, T\}, \quad \mathbb{T}_p = \{0, r^p, 2r^p, \dots, T\}, \quad \mathbb{T} = \mathbb{T}_g \cup \mathbb{T}_p. \tag{3}$$

The main advantage of using homogeneous time scales is the real-world economic interpretation: r^g and r^p represent the policymaker's degree of commitment and the public's degree of wage rigidity respectively.

4.2. **Strategies and Equilibria.** The asynchronous game on time scales will commonly have multiple Nash equilibria among which we select by subgame perfection.

Definition 1. *Any subgame perfect Nash equilibrium (SPNE) in which both players play L in all their moves on the equilibrium path will be called a* ***Ramsey SPNE****.*

In the following theorem, we denote the remainder $R := \frac{r^g}{r^p} - \left\lfloor \frac{r^g}{r^p} \right\rfloor$. It will become apparent that this constant plays an important role as it determines the exact type of dynamics.

Theorem 1. *Consider the general time inconsistency game on homogeneous time scales in which* (1) *and* (3) *hold. Then all SPNE of the game are Ramsey if and only if*

$$r^g > \overline{r^g}(R) = \begin{cases} \frac{c-d}{a-d} r^p = \frac{a-b}{a-d} r^p & \text{if } \ R = 0, \\ \frac{(1+R)(c-d)}{a-d} r^p = \frac{a-b+R(c-d)}{a-d} r^p & \text{if } \ R \in (0, \bar{R}), \\ \frac{c-d-(1-R)(a-b)}{a-d} r^p = \frac{a-b}{a-d} R r^p & \text{if } \ R \in [\bar{R}, 1), \end{cases} \tag{4}$$

where $\bar{R} = \frac{q-v}{z-x+q-v}$. In the specific time inconsistency game, in which (1) *holds,* (4) *reduces to*

$$\frac{r^g}{r^p} \in \left(\frac{3}{2}, 2\right) \cup \left(\frac{5}{2}, \infty\right). \tag{5}$$

Proof. See Libich, Stehlik [7]. □

Intuitively, in order for the policymaker to be discouraged from inflating (through the threat of high wage punishment), the policymaker's commitment must be sufficiently strong relative to the public's wage rigidity. Note that the threshold r^g is not only increasing in r^p but also a function of the type of asynchronicity and the players' payoffs.

5. HETEROGENEOUS TIME SCALES

It can be argued that the players' actions are not always deterministic and/or that the frequency of their moves vary over time or follows some random process. Heterogeneous time scales allow us to examine these cases. To do so more effectively let us normalize the time horizon to $T = r^g$ throughout the rest of this paper. As above, both g and p make a simultaneous move in the first period and all $T = r^g$ periods, but the public is also able to react to the policymaker's move in between these simultaneous moves. For comparability, we keep all the other assumptions of Section 4 unchanged.

Consider an arbitrary time scale $\mathbb{T}$ and an arbitrary nondecreasing *reaction function* of the public $f : \mathbb{T} \to [0, 1]$. To demonstrate some of the insights this section examines two special cases of interest. First, under *heterogeneous (atomistic) public* the reaction function can be interpreted as the fraction of the public which has already had the possibility of moving. Second, under the public's *probabilistic moves* the reaction function can be interpreted as a cumulative distribution function of the public's moves.

To ensure that all SPNE are Ramsey it is now sufficient to show that the g's optimal play in the first move is L regardless of the public's move in the first period (i.e. the unique best response to both L and H by the public). Therefore, the two corresponding conditions are:

$$ar^g > c\int_0^{r^g} 1 - f(t)\Delta t + d\int_0^{r^g} f(t)\Delta t, \tag{6}$$

$$dr^g > b\int_0^{r^g} 1 - f(t)\Delta t + a\int_0^{r^g} f(t)\Delta t. \tag{7}$$

Theorem 2. *Consider the general time inconsistency game in which* (1) *holds and* $f : \mathbb{T} \to [0, 1]$ *be a nondecreasing reaction function. Then all SPNE of the game are Ramsey if and only if*

$$\int_0^{r^g} f(t)\Delta t > \frac{r^g}{2}. \tag{8}$$

Proof. Apparently, using (1) the inequality (8) can be rewritten into

$$\int_0^{r^g} f(t)\Delta t > \frac{b + 2a}{2b}.$$

Multiplying both sides by b we obtain

$$
\begin{aligned}
a &> \frac{-b}{2} + b \int_0^{r^g} f(t)\Delta t \\
&= \frac{-b}{2} \int_0^{r^g} 1 - f(t)\Delta t + \frac{b}{2} \int_0^{r^g} f(t)\Delta t \\
&= c \int_0^{r^g} 1 - f(t)\Delta t + d \int_0^{r^g} f(t)\Delta t,
\end{aligned}
$$

which verifies the necessary condition (6). One can use the constraints in (1) to show that the inequalities (6) and (7) are equivalent). Obviously, one can reverse the procedure of the proof to show the other implication. □

In order to make Theorem 2 more transparent we report some corollaries and an example to improve understanding. As suggested above, we concentrate on two major cases - heterogenous public and the probabilistic models.

5.1. **Heterogenous Public.** First, let us suppose that there are N distinct groups of public (unions) $p_1, \dots, p_N$ with corresponding wage rigidities $r^{p_1}, \dots, r^{p_N}$ and sizes $s_1, \dots, s_N \in [0,1]$, which satisfy the natural assumption $\sum_{i=1}^{N} s_i = 1$. In order to restrict our attention to the policymaker's first move, we assume that r^g is a natural multiple of r^{p_i} for all $i = 1, \dots, N$, i.e.

$$\frac{r^g}{r^{p_i}} \in \mathbb{N},$$

for more asynchronous cases see [7] which shows that the intuition remains unchanged. In this special case, we can rewrite Theorem 2 in the following way.

Corollary 1. *Consider the general time inconsistency game in which* (1) *holds and public consists of N distinct groups. Then all SPNE of the game are Ramsey if and only if*

$$\sum_{i=1}^{N} s_i (r^g - r^{p_i}) \geq \frac{r^g}{2}. \tag{9}$$

Proof. It suffices to consider Theorem 2 in the case when

$$\mathbb{T} = \bigcup_{i=1}^{N} \bigcup_{j=0}^{\frac{r^g}{r^{p_i}}} \{ j r^{p_i} \}.$$

and realize that the respective players are going to react to the policymaker's first move at the time r^{p_i}. □

5.2. **Probabilistic Model.** Let us revert back to the case of one unified public to separate the effect of heterogeneous public from probabilistic moves. Depending on negotiations, the public is going to react to the policymaker's first move sometime in the interval $[a, b]$, $0 < a < b < r^g$, with a uniformly distributed probability. Then Theorem 2 becomes:

Corollary 2. *Consider the general time inconsistency game in which* (1) *holds and the public's second move takes place in interval* $[a, b]$ *with a uniformly distributed probability. Then all SPNE of the game are Ramsey if and only if*

$$\frac{b^2 - a^2}{2} \geq b - \frac{r^g}{2}. \tag{10}$$

Obviously, one can consider arbitrary combinations of above approaches, i.e. heterogeneous players with probabilistic moves, whose probability can be described by arbitrary distribution functions. Without going into detail, we only include an example to illustrate this idea.

Example 1. Consider an economy with two unions, each consisting of s_1, s_2 workers respectively such that $s_1 + s_2 = 1$. Interpret one period as a quarter (which is the frequency of releasing of main macroeconomic data). Assume that $r^g = 5$ and that the reaction times of these unions, respectively, are *at least* 1 and 3 quarters but *at most* periods 1.5 and 3.5 quarters, ie it may take them half a quarter to make a decision. Assume for simplicity that the decision made within that half a quarter follows a uniform probability distribution. Then, we have:

$$\mathbb{T}_1 := \{0\} \cup [1, 1.5] \cup \{r^g\}$$
$$\mathbb{T}_2 := \{0\} \cup [3, 3.5] \cup \{r^g\}$$

The reaction function is then $f : \mathbb{T}_1 \cup \mathbb{T}_2 \to [0, 1]$:

$$f(x) = \begin{cases} 0 & \text{if} \quad x = 0, \\ 2s_1\,(x-1) & \text{if} \quad x \in [1, 1.5]\,, \\ s_1 + 2s_2\,(x-3) & \text{if} \quad x \in [3, 3.5] \\ 1 & \text{if} \quad x = 5, \end{cases}$$

Integrating over $\mathbb{T}_1 \cup \mathbb{T}_2$ we obtain

$$\int_0^5 f(x)\Delta x = \frac{15}{4}s_1 + \frac{7}{4}s_2.$$

The assumption (8) of Theorem 2 is satisfied (and therefore the unique equilibrium is Ramsey) if and only if

$$s_1 \geq \frac{3}{8}, \text{ or equivalently, } s_2 \leq \frac{5}{8}.$$

Intuitively, the faster Union must be sufficiently large in order for the overall reaction of the public to be fast enough to discourage the policymaker from inflating.

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REFERENCES

1. R. J. Barro and D.B. Gordon, *Rules, discretion and reputation in a model of monetary policy*, Journal of Monetary Economics **12** (1983), 101–121.
2. M. Bohner and A. Peterson, *Dynamic equations on time scales*, Birkhäuser Boston Inc., Boston, MA, 2001, An introduction with applications.
3. M. Bohner and A. Peterson (eds.), *Advances in dynamic equations on time scales*, Birkhäuser Boston Inc., Boston, MA, 2003.
4. S. Hilger, *Analysis on measure chains—a unified approach to continuous and discrete calculus*, Results Math. **18** (1990), no. 1-2, 18–56.
5. F.E. Kydland and E.C. Prescott, *Rules rather than discretion: The inconsistency of optimal plans*, Journal of Political Economy **85** (1977), 473–91.
6. J. Libich, A. Hughes Hallet, and P. Stehlik, *Monetary and fiscal policy interaction with various degrees and types of commitment*, Submitted.
7. J. Libich and P. Stehlik, *Incorporating rigidity into timing structure of macroeconomic games*, Submitted.
8. E. Maskin and J. Tirole, *Theory of dynamic oligopoly, I: Overview and quantity competition with large fixed costs*, Econometrica **56** (1988), 549–569.